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Quasienergy states of trapped ions

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Abstract. The quantum models for a single trapped ion are extended to the description of the collective dynamics for systems of ions confined in quadrupole electromagnetic traps with cylindrical symmetry. A class of quantum Hamiltonians with suitable axial and radial interaction potentials given by homogeneous functions of degree (-2) and invariant under translations and axial rotations are introduced. The considered axial and radial quantum Hamiltonians for the center-of-mass and relative motions are described by collective dynamical systems associated to the symplectic group $Sp(2,\mathbb{R})$. Discrete quasienergy spectra are obtained and the corresponding quasienergy states are explicitly realized as $Sp(2,\mathbb{R})$ coherent states parameterized by the stable solutions of the corresponding classical motion equations. Consequently, a correspondence between quantum and classical stability domains is established.

PACS. 32.80.Pj Optical cooling of atoms; trapping – 03.65.Ge Solutions of wave equations: bound states

1 Introduction

It is known that ion traps [1,2] provide a single ion or a few charged particles almost at rest and practically completely isolated from the environment, permitting the study of the structure of the atomic particles, their mutual interactions and their interaction with light. In a Paul trap, regular and stable patterns with macroscopic metallic conglomerates [3] and also stable ordered structures similar to the usual atomic crystals were obtained [4–7]. The theoretical and experimental studies show that the classical trajectories of the stored ions can be regular in the quasiperiodic regime and that at very low temperatures the ions can form almost static crystalline structures. At sub-kelvin temperatures, quantum effects appear and their description requires a quantum mechanical study of stored ion dynamics. The electromagnetic ion trap has proven to be a very powerful tool for the study of fundamental quantum phenomena and the observation of Wigner crystals [8–11].

The study of the quantum dynamics of a single ion stored in a Paul trap has already been performed [12–20]. The solutions of the time-dependent Schrödinger equation for a Hamiltonian given by a second-order polynomial operator in coordinates and momenta have been obtained in [21–24]. These solutions can be applied to an ion system with harmonic interaction. Moreover, a quantum one-dimensional approach for the dynamics of a system of two or three ions inside a Paul trap was developed [25] and a complete analytic solution was obtained assuming an ion centrifugal interaction. The collective center of mass dynamics in a Paul trap have been studied in [26].

In this paper, we start an N-body quantum approach of the ion dynamics inside a Paul or combined trap considering an interaction potential given by a homogeneous function of degree (-2), invariant under translations and axial rotations. In the particular case of axial motion of N pairwise interacting particles, this potential is the Calogero potential for the one-dimensional N-body exactly solvable problem with quadratic and inversely quadratic pair potentials [27,28]. We show that the collective center of mass quantum motion [26] of the system of N stored ions is similar to the quantum one-particle dynamics in a Paul trap. Correspondingly, the quasienergy spectrum and eigenfunctions can be explicitly given according to [20]. Moreover, the axial and radial intrinsic motions can be described by the linear system for the dynamical symplectic group $Sp(2,\mathbb{R})$. Then, using group representation theory, we obtain explicit analytic solutions for the time-dependent Schrödinger equation describing the collective quantum dynamics in a Paul or combined trap.

The paper is organized as follows. In Section 2, we introduce a quantum Hamiltonian approach for systems of N ions confined in quadrupole electromagnetic traps with cylindrical symmetry. In Section 3, we show that both the axial and radial center of mass motions can be described by the quasienergy states for appropriate linear dynamical systems associated with the symplectic group $Sp(2,\mathbb{R})$. We explicitly show that the center of mass collective motion is similar to the quantum one-particle dynamics. In Section 4, we consider a suitable class of interaction

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potentials and show that both the axial and radial intrinsic motions can be also described by linear collective models associated with the dynamical group $Sp(2,\mathbb{R})$. We obtain discrete quasienergy spectra and explicit quasienergy states realized by the $Sp(2,\mathbb{R})$ coherent states parameterized by the stable solutions of the corresponding classical motion Hill equations. In Section 5, some concluding remarks are presented. In the Appendix, we consider the class of the $Sp(2,\mathbb{R})$ time-dependent linear dynamical systems and obtain the quasienergy states in terms of coherent states based on the extremal and non-extremal weight vectors. These symplectic coherent states are parameterized by the stable solutions of a Riccati equation obtained by dequantization on the Poincaré half plane considered as $Sp(2,\mathbb{R})$ phase space. The preceding results are applied to Sections 2 and 3.

2 Quantum Hamiltonians for N stored ions

In this section, we briefly discuss the Hamiltonian approach of quantum dynamics for a system of ions confined in a quadrupole electromagnetic trap with cylindrical symmetry. We consider the following quantum Hamiltonian for a system of N identical ions of mass M and electric charge Q stored in a Paul or combined trap:

$$H = \sum_{\alpha=1}^{N} H_\alpha + V, \tag{1}$$

where V is the interaction potential between ions and

$$H_\alpha = \frac{1}{2M}\left(\mathbf{p}_\alpha - \frac{1}{2}Q\,\mathbf{B}\times\mathbf{r}_\alpha\right)^2 + Q\Phi(\mathbf{r}_\alpha, t). \tag{2}$$

Here $\mathbf{r}_\alpha = (x_{\alpha1}, x_{\alpha2}, x_{\alpha3})$ and $\mathbf{p}_\alpha = (p_{\alpha1}, p_{\alpha2}, p_{\alpha3})$ are the coordinate and momentum vector operators of the particle α, $1 \le \alpha \le N$, with $p_{\alpha j} = -\mathrm{i}\hbar\partial/\partial x_{\alpha j}$, $1 \le j \le 3$. The constant axial magnetic field is given by $\mathbf{B} = (0, 0, B)$. The quadrupole electric potential is written as

$$\Phi(\mathbf{r}_\alpha, t) = A(t)(x_{\alpha1}^2 + x_{\alpha2}^2 - 2x_{\alpha3}^2), \tag{3}$$

where the function A is either time-periodic in the Paul trap case [1] or stationary in the Penning trap case [2]. For an ideal Paul trap, this function has the period $T = 2\pi/\Omega$ and is given by

$$A(t) = (r_0^2 + 2z_0^2)^{-1}(U_0 + V_0\cos\Omega t), \tag{4}$$

where U_0 and V_0 are the static and the time-varying voltages applied to the trap of semi-axes r_0 and z_0.

The Hamiltonian H can be rewritten in the form

$$H = \sum_{\alpha=1}^{N}\left[-\frac{\hbar^2}{2M}\sum_{j=1}^{3}\frac{\partial^2}{\partial x_{\alpha j}^2} + \frac{1}{2}M\lambda_{\mathrm{r}}(x_{\alpha1}^2 + x_{\alpha2}^2) + \frac{1}{2}M\lambda_{\mathrm{a}}x_{\alpha3}^2 - \frac{1}{2}\omega_{\mathrm{c}}L_{\alpha3}\right] + V, \tag{5}$$

where

$$\omega_{\mathrm{c}} = \frac{Q}{M}B,\ \lambda_{\mathrm{a}} = -4\frac{Q}{M}A(t)\ ,\ \lambda_{\mathrm{r}} = \frac{1}{4}(\omega_{\mathrm{c}}^2 - 2\lambda_{\mathrm{a}}), \tag{6}$$

and the axial angular momentum operator for the particle α is given by

$$L_{\alpha3} = x_{\alpha1}p_{\alpha2} - x_{\alpha2}p_{\alpha1}. \tag{7}$$

We now introduce the following translation-invariant coordinates $y_{\alpha j}$ and translation-invariant differential operators $D_{\alpha j}$:

$$y_{\alpha j} = x_{\alpha j} - x_j, \qquad x_j = \frac{1}{N}\sum_{\alpha=1}^{N} x_{\alpha j}, \tag{8}$$

$$D_{\alpha j} = \frac{\partial}{\partial x_{\alpha j}} - D_j\ ,\quad D_j = \frac{1}{N}\sum_{\alpha=1}^{N}\frac{\partial}{\partial x_{\alpha j}}, \tag{9}$$

where $\alpha = 1, ..., N$ and $j = 1, 2, 3$. From equations (8, 9) we find

$$\sum_{\alpha=1}^{N} y_{\alpha j} = 0,\ \sum_{\alpha=1}^{N} D_{\alpha j} = 0,\ D_{\beta k}(y_{\alpha j}) = \delta_{kj}\left(\delta_{\alpha\beta} - \frac{1}{N}\right). \tag{10}$$

From equations (7–9) we also get

$$\sum_{\alpha=1}^{N} x_{\alpha j}^2 = Nx_j^2 + \sum_{\alpha=1}^{N} y_{\alpha j}^2,\ \sum_{\alpha=1}^{N}\frac{\partial^2}{\partial x_{\alpha j}^2} = ND_j^2 + \sum_{\alpha=1}^{N} D_{\alpha j}^2, \tag{11}$$

$$\sum_{\alpha=1}^{N} L_{\alpha3} = NL_3 + L_3',\ L_3 = x_1p_2 - x_2p_1,$$

$$L_3' = -\mathrm{i}\hbar\sum_{\alpha=1}^{N}(y_{\alpha1}D_{\alpha2} - y_{\alpha2}D_{\alpha1}), \tag{12}$$

where $p_j = -\mathrm{i}\hbar D_j$, NL_3 is the center of mass angular momentum and L_3' is the intrinsic angular momentum of the N-particle system. Now, using equations (7–12) we can rewrite (5) as

$$H = H_{\mathrm{cm}} + H', \tag{13}$$

where the center-of-mass Hamiltonian is given by

$$H_{\mathrm{cm}} = N\left[\frac{1}{2M}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}M\lambda_{\mathrm{r}}(x_1^2 + x_2^2) + \frac{1}{2}M\lambda_{\mathrm{a}}x_3^2 - \frac{1}{2}\omega_{\mathrm{c}}L_3\right], \tag{14}$$

and the intrinsic Hamiltonian is defined by

$$H' = \sum_{\alpha=1}^{N}\left[-\frac{\hbar^2}{2M}\sum_{j=1}^{3} D_{\alpha j}^2 + \frac{1}{2}M\lambda_{\mathrm{r}}(y_{\alpha1}^2 + y_{\alpha2}^2) + \frac{1}{2}M\lambda_{\mathrm{a}}y_{\alpha3}^2\right] - \frac{1}{2}\omega_{\mathrm{c}}L_3' + V. \tag{15}$$

The quasienergy operator $\mathrm{i}\hbar\partial/\partial t - H$ associated with the time-periodic Hamiltonian H with period T [17,28,29] is characterized by a regular long-time behavior. A quasienergy eigenstate Ψ of definite quasienergy E is localized in space and satisfies the relation

$$\Psi(t+T) = \exp\left(-\frac{\mathrm{i}}{\hbar}ET\right)\Psi(t). \tag{16}$$

For a suitable class of interaction potentials, the Hamiltonian H has a discrete quasienergy spectrum in the stability regions of the Hill equation. In the following sections we apply this quantum stability approach to the center of mass and intrinsic collective Hamiltonians.

3 Center of mass quantum dynamics

In this section, we study the dynamic group structure and quasienergy spectrum of the center of mass Hamiltonian for a system of N stored ions. We consider the following center of mass operators:

$$\begin{aligned} K_{\mathrm{a}0} &= \frac{1}{4}(x_3^2 - D_3^2),\\ K_{\mathrm{a}\pm} &= -\frac{1}{4}(x_3^2 + D_3^2 \mp 2x_3D_3) \pm \frac{1}{2}, \end{aligned} \tag{17}$$

$$\begin{aligned} K_{\mathrm{r}0} &= \frac{1}{4}(x_1^2 + x_2^2 - D_1^2 - D_2^2),\\ K_{\mathrm{r}\pm} &= -\frac{1}{4}(x_1^2 + x_2^2 \mp 2x_1D_1 \mp 2x_2D_2) \pm 1. \end{aligned} \tag{18}$$

Using equations (17, 18), the center of mass Hamiltonian (14) becomes

$$H_{\mathrm{cm}} = N(H_{\mathrm{a}} + H_{\mathrm{r}} - \frac{1}{2}\omega_{\mathrm{c}}L_3), \tag{19}$$

$$H_{\mathrm{a}} = \alpha_{\mathrm{a}}K_{\mathrm{a}0} + \beta_{\mathrm{a}}K_{\mathrm{a}1},\ H_{\mathrm{r}} = \alpha_{\mathrm{r}}K_{\mathrm{r}0} + \beta_{\mathrm{r}}K_{\mathrm{r}1}, \tag{20}$$

where

$$K_{\mathrm{a}1} = \frac{1}{2}(K_{\mathrm{a}+} + K_{\mathrm{a}-}),\ K_{\mathrm{r}1} = \frac{1}{2}(K_{\mathrm{r}+} + K_{\mathrm{r}-}), \tag{21}$$

$$\alpha_{\mathrm{a}} = \hbar^2M^{-1} + M\lambda_{\mathrm{a}}, \quad \beta_{\mathrm{a}} = \hbar^2M^{-1} - M\lambda_{\mathrm{a}}, \tag{22}$$

$$\alpha_{\mathrm{r}} = \hbar^2M^{-1} + M\lambda_{\mathrm{r}}, \quad \beta_{\mathrm{r}} = \hbar^2M^{-1} - M\lambda_{\mathrm{r}}. \tag{23}$$

The center of mass dynamical group G_{cm} is the direct product $G_{\mathrm{cm}} = G_{\mathrm{a}}\otimes G_{\mathrm{r}}\otimes SO(2)$, where the axial symplectic group G_{a} has the infinitesimal generators $K_{\mathrm{a}0}$, $K_{\mathrm{a}1}$, $K_{\mathrm{a}2} = \mathrm{i}[K_{\mathrm{a}0}, K_{\mathrm{a}1}]$, the radial symplectic group G_{r} has the infinitesimal generators $K_{\mathrm{r}0}$, $K_{\mathrm{r}1}$, $K_{\mathrm{r}2} = \mathrm{i}[K_{\mathrm{r}0}, K_{\mathrm{r}1}]$, and the infinitesimal generator of the axial rotation group $SO(2)$ is L_3. The groups G_{a} and G_{r} are isomorphic to the symplectic group $Sp(2,\mathbb{R})$. A unitary irreducible representation of G is labeled by the Bargmann index k_{a} for G_{a}, where either $k_{\mathrm{a}} = 1/4$, or $k_{\mathrm{a}} = 3/4$, and by the Bargmann index $k_{\mathrm{r}} = (l+1)/2$ for G_{r}, where l is a non-negative integer. Then the axial and radial Hamiltonians (20) describe two time-dependent linear dynamical systems for $Sp(2,\mathbb{R})$ of the form (A.1) with $a = \alpha_{\mathrm{a}}$, $b = \beta_{\mathrm{a}}$, $c = 0$ and $a = \alpha_{\mathrm{r}}$, $b = \beta_{\mathrm{r}}$, $c = 0$, respectively.

Using (A.15), the quasienergy eigenvectors of the quasienergy operator $\mathrm{i}\hbar\partial/\partial t - H_{\mathrm{cm}}$ with definite quasienergies $E_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l}$ are obtained in the form

$$\begin{aligned} \Psi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l} = \exp\left[-\mathrm{i}(k_{\mathrm{a}} + m_{\mathrm{a}})\varphi_{\mathrm{a}} - \mathrm{i}(k_{\mathrm{r}} + m_{\mathrm{r}})\varphi_{\mathrm{r}}\right]&\\ \times U_{\mathrm{a}}(z_{\mathrm{a}})U_{\mathrm{r}}(z_{\mathrm{r}})\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l},& \end{aligned} \tag{24}$$

where the vectors $\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l}$ with m_{a} and m_{r} non-negative integers are satisfying the equations

$$\begin{aligned} K_{\mathrm{a}}\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l} &= (k_{\mathrm{a}} + m_{\mathrm{a}})\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l},\\ K_{\mathrm{r}}\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l} &= (k_{\mathrm{r}} + m_{\mathrm{r}})\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l},\\ K_{\mathrm{a}-}\Phi_{k_{\mathrm{a}}0,k_{\mathrm{r}}\,m_{\mathrm{r}},l} &= K_{\mathrm{r}-}\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,0,l} = 0,\\ L_3\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l} &= \hbar l\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l}. \end{aligned} \tag{25}$$

According to (A.6), the operators $U_{\mathrm{a}}(z_{\mathrm{a}})$ and $U_{\mathrm{r}}(z_{\mathrm{r}})$ are given by

$$U_{\mathrm{a}}(z_{\mathrm{a}}) = \exp\left(z_{\mathrm{a}}K_{\mathrm{a}+}\right)\exp\left(\eta_{\mathrm{a}}K_{\mathrm{a}0}\right)\exp\left(-\bar{z}_{\mathrm{a}}K_{\mathrm{a}-}\right), \tag{26}$$

$$U_{\mathrm{r}}(z_{\mathrm{r}}) = \exp\left(z_{\mathrm{r}}K_{\mathrm{r}+}\right)\exp(\eta_{\mathrm{r}}K_{\mathrm{r}0})\exp\left(-\bar{z}_{\mathrm{r}}K_{\mathrm{r}-}\right), \tag{27}$$

where $\eta_{\mathrm{a}} = \ln(1 - z_{\mathrm{a}}\bar{z}_{\mathrm{a}})$ and $\eta_{\mathrm{r}} = \ln(1 - z_{\mathrm{r}}\bar{z}_{\mathrm{r}})$. Using the differential equation (A.11) and denoting $\tau = N^{-1}t$, we obtain

$$\begin{aligned} \hbar\frac{\mathrm{d}\varphi_{\mathrm{a}}}{\mathrm{d}\tau} &= \alpha_{\mathrm{a}} + \frac{1}{2}\beta_{\mathrm{a}}(z_{\mathrm{a}} + \bar{z}_{\mathrm{a}}),\\ \hbar\frac{\mathrm{d}\varphi_{\mathrm{r}}}{\mathrm{d}\tau} &= \alpha_{\mathrm{r}} + \frac{1}{2}\beta_{\mathrm{r}}(z_{\mathrm{r}} + \bar{z}_{\mathrm{r}}). \end{aligned} \tag{28}$$

According to equations (A.16, A.14), the complex coordinates z_{a} and z_{r} can be written

$$z_{\mathrm{a}} = \left[M\lambda_{\mathrm{a}}u_{\mathrm{a}} + \mathrm{i}\hbar\frac{\mathrm{d}u_{\mathrm{a}}}{\mathrm{d}\tau}\right]\left[M\lambda_{\mathrm{a}}u_{\mathrm{a}} - \mathrm{i}\hbar\frac{\mathrm{d}u_{\mathrm{a}}}{\mathrm{d}\tau}\right]^{-1}, \tag{29}$$

$$z_{\mathrm{r}} = \left[M\lambda_{\mathrm{a}}u_{\mathrm{r}} + \mathrm{i}\hbar\frac{\mathrm{d}u_{\mathrm{r}}}{\mathrm{d}\tau}\right]\left[M\lambda_{\mathrm{a}}u_{\mathrm{r}} - \mathrm{i}\hbar\frac{\mathrm{d}u_{\mathrm{r}}}{\mathrm{d}\tau}\right]^{-1}, \tag{30}$$

where u_{a} and u_{r} are stable solutions of the Hill equations

$$\frac{\mathrm{d}^2u_{\mathrm{a}}}{\mathrm{d}\tau^2} + \lambda_{\mathrm{a}}u_{\mathrm{a}} = 0,\ \frac{\mathrm{d}^2u_{\mathrm{r}}}{\mathrm{d}\tau^2} + \lambda_{\mathrm{r}}u_{\mathrm{r}} = 0. \tag{31}$$

The center of mass quasienergies corresponding to $\Psi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l}$ can be written as

$$E_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l} = 2\mu_{\mathrm{a}}(k_{\mathrm{a}} + m_{\mathrm{a}}) + 2\mu_{\mathrm{r}}(k_{\mathrm{r}} + m_{\mathrm{r}}) - \frac{1}{2}\omega_{\mathrm{c}}l, \tag{32}$$

where μ_{a} and μ_{r} are the Floquet exponents of u_{a} and u_{r}. Some explicit realizations of the quasienergy eigenvectors in the case of single ion dynamics have been obtained in [17,20]. These realizations, with time replaced by τ, can be applied to the center of mass Hamiltonian.

4 Collective intrinsic dynamics

In this section, we consider a class of axial and radial interaction potentials given by translation-invariant homogeneous functions of degree (-2), invariant under the intrinsic axial rotations, and show that the axial and radial intrinsic dynamics can be described by linear collective models associated with the dynamical group $Sp(2,\mathbb{R})$. We obtain discrete quasienergy spectra and explicit quasienergy states realized by the $Sp(2,\mathbb{R})$ coherent states parameterized by the stable solutions of the corresponding Hill classical motion equations.

We introduce the following operators:

$$K'_{\mathrm{a}0} = \frac{1}{4}\sum_{\alpha=1}^{N}(y^2_{\alpha 3} - D^2_{\alpha 3}) + W_{\mathrm{a}},$$
$$K'_{\mathrm{a}\pm} = -\frac{1}{4}\sum_{\alpha=1}^{N}(y^2_{\alpha 3} + D^2_{\alpha 3} \mp 2y_{\alpha 3}D_{\alpha 3}) \pm \frac{1}{2}(N-1) + W_{\mathrm{a}}, \tag{33}$$

$$K'_{\mathrm{r}0} = \frac{1}{4}\sum_{\alpha=1}^{N}\sum_{j=1}^{2}(y^2_{\alpha j} - D^2_{\alpha j}) + W_{\mathrm{r}},$$
$$K'_{\mathrm{r}\pm} = -\frac{1}{4}\sum_{\alpha=1}^{N}\sum_{j=1}^{2}(y^2_{\alpha j} + D^2_{\alpha j} \mp 2y_{\alpha j}D_{\alpha j}) \pm (N-1) + W_{\mathrm{r}}, \tag{34}$$

where the axial potential W_{a} is a function of $y_{\alpha 3}$ $(1 \leq \alpha \leq N)$, and the radial potential W_{r} is a function of $y_{\alpha 1}$ and $y_{\alpha 2}$ $(1 \leq \alpha \leq N)$ such that $[L'_3, W_{\mathrm{r}}] = 0$. Moreover, we suppose that W_{a} and W_{r} are homogeneous functions of degree (-2). Then the Euler theorem gives

$$\sum_{\alpha=1}^{N} y_{\alpha 3}D_{\alpha 3}(W_{\mathrm{a}}) = -2W_{\mathrm{a}},$$
$$\sum_{\alpha=1}^{N}\sum_{j=1}^{2} y_{\alpha j}D_{\alpha j}(W_{\mathrm{r}}) = -2W_{\mathrm{r}}. \tag{35}$$

A particular axial potential is considered for the one-dimensional N-body exactly solvable Calogero dynamical system with quadratic and inversely quadratic pair potentials [27,28]:

$$W_{\mathrm{a}} = g^2 \sum_{1<\alpha<\beta<N} \frac{1}{(y_{\alpha j} - y_{\beta j})^2}\,. \tag{36}$$

According to (35), the operators (33, 34) satisfy the commutation relations for the Lie algebra of the symplectic group $Sp(2,\mathbb{R})$:

$$[K'_{\mathrm{a}-}, K'_{\mathrm{a}+}] = 2K'_{\mathrm{a}0},\ [K'_{\mathrm{a}0}, K'_{\mathrm{a}\pm}] = \pm K'_{\mathrm{a}\pm}, \tag{37}$$
$$[K'_{\mathrm{r}-}, K'_{\mathrm{r}+}] = 2K'_{\mathrm{r}0},\ [K'_{\mathrm{r}0}, K'_{\mathrm{r}\pm}] = \pm K'_{\mathrm{r}\pm}. \tag{38}$$

We remark that the axial operators (33) commute with the radial operators (34). Moreover, the angular momentum L'_3 commutes with either operator from (33, 34). We introduce the axial symplectic group G'_{a} with the infinitesimal generators $K'_{\mathrm{a}0}$, $K'_{\mathrm{a}1}$, $K'_{\mathrm{a}2}$ and the radial symplectic group G'_{r} with the infinitesimal generators $K'_{\mathrm{r}0}$, $K'_{\mathrm{r}1}$, $K'_{\mathrm{r}2}$, where

$$K'_{\mathrm{a}1} = \frac{1}{2}(K'_{\mathrm{a}+} + K'_{\mathrm{a}-}),\ K'_{\mathrm{a}2} = \frac{\mathrm{i}}{2}(K'_{\mathrm{a}-} - K'_{\mathrm{a}+}), \tag{39}$$
$$K'_{\mathrm{r}1} = \frac{1}{2}(K'_{\mathrm{r}+} + K'_{\mathrm{r}-}),\ K'_{\mathrm{r}2} = \frac{\mathrm{i}}{2}(K'_{\mathrm{r}-} - K'_{\mathrm{r}+}). \tag{40}$$

The intrinsic groups G'_{a} and G'_{r} are isomorphic to the symplectic group $Sp(2,\mathbb{R})$. The intrinsic dynamical group is the direct product $G' = G'_{\mathrm{a}} \otimes G'_{\mathrm{r}} \otimes R'$, where the infinitesimal generator of the intrinsic axial rotation group R' is L'_3.

Using equations (15, 33, 34), the intrinsic Hamiltonian can be written as

$$H' = H'_0 + V', \quad H'_0 = H'_{\mathrm{a}} + H'_{\mathrm{r}} - \frac{1}{2}\omega_{\mathrm{c}}L'_3, \tag{41}$$
$$V' = V - 2\hbar^2 M^{-1}(W_{\mathrm{a}} + W_{\mathrm{r}}), \tag{42}$$
$$H'_{\mathrm{a}} = \alpha_{\mathrm{a}}K'_{\mathrm{a}0} + \beta_{\mathrm{a}}K'_{\mathrm{a}1},\ H'_{\mathrm{r}} = \alpha_{\mathrm{r}}K'_{\mathrm{r}0} + \beta_{\mathrm{r}}K'_{\mathrm{r}1}, \tag{43}$$

where α_{a}, β_{a}, α_{r} and β_{r} are given by (22, 23). Then the axial and radial intrinsic Hamiltonians (43) describe two linear dynamical systems for $Sp(2,\mathbb{R})$ of the form (A.1) with $a = \alpha_{\mathrm{a}}$, $b = \beta_{\mathrm{a}}$, $c = 0$ and $a = \alpha_{\mathrm{r}}$, $b = \beta_{\mathrm{r}}$, $c = 0$, respectively. If $V' = 0$, then the intrinsic dynamics can be described by the linear Hamiltonian H'_0 for the intrinsic dynamical group G'. A unitary irreducible representation of G' is labeled by three non-negative integers n_{a}, n_{r} and l. The Bargmann index k_{a} of G'_{a} and the Bargmann index k_{r} of G'_{r} are given by

$$k_{\mathrm{a}} = \frac{1}{4}(N-1) + \frac{1}{2}n_{\mathrm{a}}, \quad k_{\mathrm{r}} = \frac{1}{2}(N-1) + \frac{1}{2}n_{\mathrm{r}}. \tag{44}$$

Using (A.15), the quasienergy eigenvectors of the quasienergy operator $\mathrm{i}\hbar\partial/\partial t - H'_0$ are obtained in the form

$$\Psi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s} = \exp\left[-\mathrm{i}(k_{\mathrm{a}} + m_{\mathrm{a}})\varphi_{\mathrm{a}} - \mathrm{i}(k_{\mathrm{r}} + m_{\mathrm{r}})\varphi_{\mathrm{r}}\right] \times U'_{\mathrm{a}}(z_{\mathrm{a}})U'_{\mathrm{r}}(z_{\mathrm{r}})\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s}, \tag{45}$$

where the basis vectors $\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s}$ are satisfying the equations

$$K'_{\mathrm{a}0}\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s} = (k_{\mathrm{a}} + m_{\mathrm{a}})\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s},$$
$$K'_{\mathrm{r}0}\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s} = (k_{\mathrm{r}} + m_{\mathrm{r}})\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s},$$
$$K'_{\mathrm{a}-}\Phi_{k_{\mathrm{a}}\,0,k_{\mathrm{r}}\,m_{\mathrm{r}},l,s} = K'_{\mathrm{r}-}\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,0_{\mathrm{r}},l,s} = 0,$$
$$L'_3\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s} = \hbar l\Phi_{k_{\mathrm{a}}\,m_{\mathrm{a}},k_{\mathrm{r}}\,m_{\mathrm{r}},l,s}, \tag{46}$$

for m_{a}, $m_{\mathrm{r}} = 0, 1, \ldots$ Here s is a degeneracy index. In the particular case of the Calogero potential, the explicit expressions for the degeneracy generating functions are presented in [28].

The unitary operators $U'_{\mathrm{a}}(z_{\mathrm{a}})$ and $U'_{\mathrm{r}}(z_{\mathrm{r}})$ are defined by (A.6):

$$U'_{\mathrm{a}}(z_{\mathrm{a}}) = \exp(z_{\mathrm{a}}K'_{\mathrm{a}+})\exp(\eta_{\mathrm{a}}K'_{\mathrm{a}0})\exp(-\bar{z}_{\mathrm{a}}K'_{\mathrm{a}-}), \tag{47}$$
$$U'_{\mathrm{r}}(z_{\mathrm{r}}) = \exp(z_{\mathrm{r}}K'_{\mathrm{r}+})\exp(\eta_{\mathrm{r}}K_{\mathrm{r}0})\exp(-\bar{z}_{\mathrm{r}}K'_{\mathrm{r}-}), \tag{48}$$

where $\eta_a = \ln(1 - z_a\bar{z}_a)$ and $\eta_r = \ln(1 - z_r\bar{z}_r)$.

The complex coordinates z_a and z_r, phases φ_a and φ_r and the quasienergy spectrum are given by (28–32) with τ replaced by t

$$\hbar\frac{d\varphi_a}{dt} = \alpha_a + \frac{1}{2}\beta_a(z_a + \bar{z}_a),$$
$$\hbar\frac{d\varphi_r}{dt} = \alpha_r + \frac{1}{2}\beta_r(z_r + \bar{z}_r). \qquad (49)$$

According to (A.16, A.14), the complex coordinates z_a and z_r can be written

$$z_a = \left[M\lambda_a u_a + i\hbar\frac{du_a}{dt}\right]\left[M\lambda_a u_a - i\hbar\frac{du_a}{dt}\right]^{-1}, \qquad (50)$$

$$z_r = \left[M\lambda_a u_r + i\hbar\frac{du_r}{dt}\right]\left[M\lambda_a u_r - i\hbar\frac{du_r}{dt}\right]^{-1}, \qquad (51)$$

$$\frac{d^2u_a}{dt^2} + \lambda_a u_a = 0,\ \frac{d^2u_r}{dt^2} + \lambda_r u_r = 0, \qquad (52)$$

$$E_{k_a\,m_a,k_r\,m_r,l,s} = 2\mu_a(k_a + m_a) + 2\mu_r(k_r + m_r) - \frac{1}{2}\omega_c l, \qquad (53)$$

where μ_a and μ_r are the Floquet exponents of the stable solutions u_a and u_r of the Hill equations (52). The intrinsic quasienergies corresponding to quasienergy eigenvectors $\Psi_{k_a\,m_a,k_r\,m_r,l}, s$ are given by (53).

5 Concluding remarks

In this paper we have developed a collective quantum model for ion systems confined in quadrupole electromagnetic traps with cylindrical symmetry. This model describes the collective motion of stored ions observed in [26]. We have considered suitable axial and radial interaction potentials given by translation-invariant homogeneous functions of degree (-2), invariant under axial rotations. We have shown that both axial and radial dynamics of the center of mass and of the intrinsic motion of a system of N stored ions can be described by the linear collective models of the dynamical group $Sp(2,\mathbb{R})$. Then the properties of the dynamical group and of the associated coherent states are used in order to obtain analytic exact solutions of the time-dependent Schrödinger equation, characterizing the collective quantum dynamics of ions stored in a Paul or a combined trap. These solutions describe the crystalline states appearing on the symmetry axis and in the symmetry plane.

We have obtained discrete quasienergy spectra and explicit quasienergy eigenstates realized by coherent states for the dynamical symplectic group $Sp(2,\mathbb{R})$. These symplectic coherent states are parameterized by the stable solutions of a classical motion Hill equation, obtained by dequantization on the Poincaré half plane classically considered phase space. This result follows from the Berezin quantization [28] and the time-dependent variational principle [30]. The system of quasienergy states describes a quantum crystallization regime. No phase transition was obtained, because the collective models are integrable and have discrete quasienergy spectra. Introducing specific Coulomb interactions and trap anharmonicities that give a coupling between the radial and axial degrees of freedom, continuous and nonlinear discrete quasienergy spectra can be obtained for some control parameters corresponding to the classical chaotic regimes. Then suitable effective interactions realized as biparticle potentials multiplied by functions of collective coordinates can be introduced. We will concern ourselves with these aspects in the next paper [31].

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Appendix

In this appendix we study the time-dependent linear dynamical systems associated with the dynamical symplectic group $Sp(2,\mathbb{R})$ and obtain the quasienergy eigenstates in terms of coherent states parameterized by the stable solutions of the classical motion equations obtained by dequantization on the Poincaré half plane. We consider a quantum linear $Sp(2,\mathbb{R})$ system described by the Hamiltonian

$$H = aK_0 + bK_1 + cK_2, \qquad (A.1)$$

where a, b and c are time-dependent functions. The basis of the Lie algebra of $Sp(2,\mathbb{R})$ consists of three generators K_0, K_1 and K_2, such that the raising and lowering operators $K_\pm = K_1 \pm i\,K_2$ satisfy the following commutation relations:

$$[K_0, K_\pm] = \pm K_\pm,\ [K_-, K_+] = 2K_0. \qquad (A.2)$$

The Casimir operator

$$C_2 = K_0^2 - K_1^2 - K_2^2, \qquad (A.3)$$

has eigenvalues denoted by $k(k-1)$, where k is the Bargmann index for unitary irreducible representations of $Sp(2,\mathbb{R})$ [28]. We consider only the positive discrete series of $Sp(2,\mathbb{R})$ representations. For the lowest weight unitary irreducible $Sp(2,\mathbb{R})$ representation space with a fixed Bargmann index $k > 0$, we consider the canonical basis consisting of the orthonormal vectors Φ_{km} for $m = 0, 1, ...$, where K_0 is diagonal and the action of K_+ and K_- is given by

$$K_+\Phi_{km} = [(m+1)(m+2k)]^{\frac{1}{2}}\Phi_{k,m+1}\,,$$
$$K_-\Phi_{km} = [m(m+2k-1)]^{\frac{1}{2}}\Phi_{k,m-1},$$
$$K_0\Phi_{km} = (k+m)\Phi_{km}. \qquad (A.4)$$

We now introduce the following coherent states for $Sp(2,\mathbb{R})$:

$$\Phi_{km}(z) = U(z)\Phi_{km}(0), \qquad (A.5)$$

where $\Phi_{km}(0) = \Phi_{km}$ and

$$U(z) = \exp(zK_+)\exp(\beta K_0)\exp(-\bar{z}K_-) \quad (A.6)$$

are group representation unitary operators with $|z| < 1$ and $\beta = \ln(1 - z\bar{z})$ [20]. For the case $m = 0$, the standard geometrical construction of extremal generalized coherent states is obtained [28]. The control parameter space is the unit disc $|z| < 1$ endowed with the Lobachevsky metrics $\mathrm{d}s^2 = 4(1 - z\bar{z})^{-2}\mathrm{d}z\mathrm{d}\bar{z}$ [28]. According to (A.5), the non-extremal coherent states are constructed by the application of the unitary operators (A.6) on the non-extremal weight vectors $\Phi_{km},\ m = 1, 2, ...$

The vector

$$\Psi_{km} = \exp(-\mathrm{i}\varphi_{km})\Phi_{km}(z), \quad (A.7)$$

evolves according to the Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Psi_{km} = H\Psi_{km}, \quad (A.8)$$

where the complex coordinate z and the phase φ_{km} are time-dependent functions satisfying the differential equations

$$\hbar\frac{\mathrm{d}\varphi_{km}}{\mathrm{d}t}(1 - z\bar{z})\Phi_{km} = \Bigg\{\left[\mathrm{i}\hbar\left(z\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t}\bar{z}\right) + a(1 + z\bar{z}) + b(z + \bar{z}) + \mathrm{i}c(z - \bar{z})\right]K_0 + \left[\mathrm{i}\hbar\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} + a\bar{z} + \frac{b}{2}(1 + \bar{z}^2) - \frac{\mathrm{i}c}{2}(\bar{z}^2 - 1)\right]K_- - \left[\mathrm{i}\hbar\frac{\mathrm{d}z}{\mathrm{d}t} - az - \frac{b}{2}(z^2 + 1) - \frac{\mathrm{i}c}{2}(z^2 - 1)\right]K_+\Bigg\}\Phi_{km}. \quad (A.9)$$

Then (A.9) holds if $\varphi_{km} = (k + m)\varphi$ and the time-dependent functions z and φ are given by the differential equations

$$\mathrm{i}\hbar\frac{\mathrm{d}z}{\mathrm{d}t} = az + \frac{b}{2}(z^2 + 1) + \frac{\mathrm{i}c}{2}(z^2 - 1) \quad (A.10)$$

$$\hbar\frac{\mathrm{d}\varphi}{\mathrm{d}t} = a + \frac{b}{2}(z + \bar{z}) + \frac{\mathrm{i}c}{2}(z - \bar{z}). \quad (A.11)$$

The unit disc $|z| < 1$ can be mapped onto the Poincaré half plane Im $w > 0$ by the Cayley transformation $z = (\mathrm{i} - w)(\mathrm{i} + w)^{-1}$. Then the motion equation (A.10) for z reduces to the following Riccatti equation:

$$\hbar\frac{\mathrm{d}w}{\mathrm{d}t} = w^2(b - a) + 2cw - a - b. \quad (A.12)$$

If $a \neq b$, then (A.12) can be linearized by substituting

$$w = \frac{2\hbar}{b - a}\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}t}, \quad (A.13)$$

where u satisfies the linear differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 2c\frac{\mathrm{d}u}{\mathrm{d}t} + fu = 0, \quad f = (2\hbar)^{-2}(b^2 - a^2). \quad (A.14)$$

Then the quasienergy solutions of the Schrödinger equation (A.8) can be written as

$$\Psi_{km} = \exp(zK_+)\exp(\beta K_0)\exp(-\bar{z}K_-)\exp(-\varphi K_0)\Phi_{km}, \quad (A.15)$$

where φ is obtained from (A.11) and

$$z = \left[\mathrm{i}(b - a)u - 2\hbar\frac{\mathrm{d}u}{\mathrm{d}t}\right]\left[\mathrm{i}(b - a)u + 2\hbar\frac{\mathrm{d}u}{\mathrm{d}t}\right]^{-1} \quad (A.16)$$

with u given by (A.14).

If $c = 0$ and f is a time-periodic function, then (A.14) is a Hill equation. The equation (A.14) for the Paul potential (4) is the standard Mathieu equation. Moreover, the quasienergy spectrum is given by

$$E_{km} = 2\hbar\mu(k + m), \quad m = 0, 1, ..., \quad (A.17)$$

where μ is the Floquet exponent for the solution u of (A.14).

It is convenient to denote

$$\hat{A} = U(-z)\,AU(z)\,,\ \ \tilde{A} = (\Psi_{k0}, A\Psi_{k0}) \quad (A.18)$$

for any polynomial operator A in K_0, K_1 and K_2. If A is a quantum observable, then the corresponding classical observable $\tilde{A}$ is given by the expectation value of A in the coherent states Ψ_{k0}. Then

$$\tilde{A} = (\Phi_{k0}, \hat{A}\Phi_{k0}). \quad (A.19)$$

For the generators K_0, K_+ and K_-, we obtain

$$\hat{K}_- = (1 - z\bar{z})^{-1}\left[K_- + 2zK_0 + z^2K_+\right] = (\hat{K}_+)^\dagger, \quad (A.20)$$

$$\hat{K}_0 = (1 - z\bar{z})^{-1}\left[\bar{z}K_- + (1 + z\bar{z})\,K_0 + zK_+\right], \quad (A.21)$$

$$\widetilde{K_0} = k\frac{1 + z\bar{z}}{1 - z\bar{z}},\ \widetilde{K_+} = k\frac{2\bar{z}}{1 - z\bar{z}},\ \widetilde{K_-} = k\frac{2z}{1 - z\bar{z}}\cdot \quad (A.22)$$

The classical Hamiltonian $\tilde{H}$ corresponding to the quantum linear system (A.1) can be written as

$$\tilde{H} = (1 - z\bar{z})^{-1}\left[a(1 + z\bar{z}) + b(z + \bar{z}) + \mathrm{i}c(\bar{z} - z)\right]. \quad (A.23)$$

Then (A.10) is exactly the classical motion equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left\{z, \tilde{H}\right\} \quad (A.24)$$

with the Poisson bracket

$$\{f, g\} = \frac{(1 - z\bar{z})^2}{2\mathrm{i}k}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}}\frac{\partial g}{\partial z}\right) \quad (A.25)$$

obtained from the Berezin quantization [28] and the time-dependent variational principle [30].

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